

MATCHING GROUPS AND GLIDING SYSTEMS

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ABSTRACT. With every matching in a graph we associate a group called the matching group. We study this group using the theory of nonpositively curved cubed complexes. Our approach is formulated in terms of so-called gliding systems.

1. INTRODUCTION

Consider a graph Γ without loops but possibly with multiple edges. A matching A in Γ is a set of edges of Γ such that different edges in A have no common vertices. Matchings are extensively studied in graph theory usually with the view to define numerical invariants of graphs. In this paper we study transformations of matchings determined by even cycles. An even cycle in Γ is an embedded circle in Γ formed by an even number of edges. If a matching A meets an even cycle s at every second edge of s , then removing these edges from A and adding instead all the other edges of s we obtain a new matching denoted sA . We say that sA is obtained from A by *gliding* along s . The inverse transformation is the gliding of sA along s which, obviously, gives back A . Composing the glidings, we can pass back and forth between matchings. If two even cycles s, t have no common vertices and a matching A meets both s and t at every second edge, then the compositions $A \mapsto sA \mapsto tsA$ and $A \mapsto tA \mapsto stA = tsA$ are considered as the same transformation. For any matching A in Γ , the compositions of glidings carrying A to itself form a group $\pi_A = \pi_A(\Gamma)$ called the *matching group*. Similar groups were first considered in [STCR] in the context of domino tilings of planar regions.

In the rest of the introduction, we focus on matching groups in finite graphs. We prove that they are torsion-free, residually nilpotent, residually finite, biorderable, biautomatic, have solvable word and conjugacy problems, satisfy the Tits alternative, embed in $SL_n(\mathbb{Z})$ for some n , and embed in finitely generated right-handed Artin groups. Our main tool in the proof of these properties is an interpretation of the matching groups as the fundamental groups of nonpositively curved cubed complexes. The universal coverings of such complexes are Cartan-Alexandrov-Toponogov (0)-spaces in the sense of Gromov (CAT(0)-spaces). All necessary definitions from the theory of cubed complexes are recalled in the paper.

Using much more elementary considerations, we give a presentation of the matching group by generators and relations as follows. The set of vertices of a finite graph Γ adjacent to the edges of a matching A in Γ is denoted ∂A . We say that two matchings A, B in Γ are *congruent* if $\partial A = \partial B$. We explain that any tuple of matchings in Γ congruent to a given matching A_0 determines an element in π_{A_0} . The group π_{A_0} is generated by the elements $\{x_{A,B}\}_{A,B}$ associated with the 2-tuples A, B of matchings congruent to A_0 . The defining relations: $x_{A_0,A} = 1$ for any A congruent to A_0 and $x_{A,C} = x_{A,B} x_{B,C}$ for any matchings A, B, C congruent to A_0

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such that every vertex in ∂A_0 is incident to an edge which belongs to at least two of the matchings A, B, C . As a consequence, the group π_{A_0} is finitely generated and its rank is smaller than or equal to $M(M-1)/2$ where M is the number of matchings in Γ congruent to A_0 and distinct from A_0 .

We define two families of natural homomorphisms between matching groups. First, any subset A' of a matching A in Γ is itself a matching in Γ . We define a canonical injection $\pi_{A'} \hookrightarrow \pi_A$. Identifying $\pi_{A'}$ with its image, one can treat $\pi_{A'}$ as a subgroup of π_A . Second, any two congruent matchings A, B in Γ may be related by glidings, and, as a consequence, their matching groups are isomorphic. We exhibit a canonical isomorphism $\pi_A \approx \pi_B$. We also relate the matching groups to the braid groups of graphs. This allows us to derive braids in graphs from tuples of matchings. We will briefly discuss a generalization of the matching groups to hypergraphs.

A special role in the theory of matchings is played by perfect matchings also called dimer coverings. A matching in a graph is perfect if every vertex of the graph is incident to a (unique) edge of this matching. Perfect matchings have been extensively studied in connection with exactly solvable models of statistical mechanics and with path algebras, see [Bo], [Ke] and references therein. The matching groups associated with perfect matchings are called *dimer groups*. Since all perfect matchings in a finite graph are congruent, their dimer groups are isomorphic. The resulting isomorphism class of groups is an invariant of the graph.

The study of glidings suggests a more general framework of gliding systems in groups. A *gliding system* in a group G consists of certain elements of G called *glides* and a relation on the set of glides called *independence* satisfying a few axioms. Given a gliding system in G and a set $\mathcal{D} \subset G$, we construct a cubed complex $X_{\mathcal{D}}$ called the *glide complex*. The fundamental groups of the components of $X_{\mathcal{D}}$ are the *glide groups*. We formulate conditions ensuring that $X_{\mathcal{D}}$ is nonpositively curved. One can view gliding systems as devices producing nonpositively curved complexes and interesting groups. The matching groups and, in particular, the dimer groups are instances of glide groups for appropriate G and \mathcal{D} .

The paper is organized as follows. In Section 2 we recall the basics on cubed complexes and cubic maps. The next three sections deal with glidings: we define the gliding systems (Section 3), construct the glide complexes (Section 4), and study natural maps between the glide groups (Section 5). Next, we introduce dimer groups (Section 6), compute them via generators and relations (Section 7), and define and study the matching groups (Section 8). In Section 9 we consider connections with braid groups. In Section 10 we interpret the dimer complex in terms of graph labelings. In Section 11 we discuss the matching groups of hypergraphs. In the appendix we examine the typing homomorphisms of the matching groups.

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2. PRELIMINARIES ON CUBED COMPLEXES AND CUBICAL MAPS

We discuss the basics of the theory of cubed complexes and cubical maps, see [BH], Chapters I.7 and II.5 for more details.

2.1. Cubed complexes. Set $I = [0, 1]$. A *cubed complex* is a CW-complex X such that each (closed) k -cell of X with $k \geq 0$ is a continuous map from the k -dimensional cube I^k to X whose restriction to the interior of I^k is injective and

whose restriction to each $(k-1)$ -face of I^k is an isometry of that face onto I^{k-1} composed with a $(k-1)$ -cell $I^{k-1} \rightarrow X$ of X . The k -cells $I^k \rightarrow X$ are not required to be injective. The k -skeleton $X^{(k)}$ of X is the union of the images of all cells of dimension $\leq k$.

For example, the cube I^3 together with all its faces is a cubed complex. So is the k -dimensional torus obtained by identifying opposite faces of I^k .

The *link* $LK(A) = LK(A; X)$ of a 0-cell A of a cubed complex X is the space of all directions at A . Each triple $(k \geq 1, \text{ a vertex } a \text{ of } I^k, \text{ a } k\text{-cell } \alpha : I^k \rightarrow X \text{ of } X \text{ carrying } a \text{ to } A)$ determines a $(k-1)$ -dimensional simplex in $LK(A)$ in the obvious way. The faces of this simplex are determined by the restrictions of α to the faces of I^k containing a . The simplices corresponding to all triples (k, a, α) cover $LK(A)$ but may not form a simplicial complex. We say, following [HaW], that the cubed complex X is *simple* if the links of all $A \in X^{(0)}$ are simplicial complexes, i.e., all simplices in $LK(A)$ are embedded and the intersection of any two simplices in $LK(A)$ is a common face.

A *flag complex* is a simplicial complex such that any finite collection of pairwise adjacent vertices spans a simplex. A cubed complex is *nonpositively curved* if it is simple and the link of each 0-cell is a flag complex. A theorem of M. Gromov asserts that the universal covering of a connected finite-dimensional nonpositively curved cubed complex is a CAT(0)-space. Since CAT(0)-spaces are contractible, all higher homotopy groups of such a complex X vanish while the fundamental group $\pi = \pi_1(X)$ is torsion-free. This group satisfies a strong form of the Tits alternative: each subgroup of π contains a rank 2 free subgroup or virtually is a finitely generated abelian group, see [SW]. Also, π does not have Kazhdan's property (T), see [NR1]. If X is compact, then π has solvable word and conjugacy problems and is biautomatic, see [NR2].

2.2. Cubical maps. A *cubical map* between simple cubed complexes X and Y is a continuous map $X \rightarrow Y$ whose composition with any k -cell $I^k \rightarrow X$ of X expands as the composition of a self-isometry of I^k with a k -cell $I^k \rightarrow Y$ of Y for all $k \geq 0$. For any $A \in X^{(0)}$, a cubical map $f : X \rightarrow Y$ induces a simplicial map $f_A : LK(A) \rightarrow LK(f(A))$. A cubical map $f : X \rightarrow Y$ is a *local isometry* if for all $A \in X^{(0)}$, the map f_A is an embedding and its image is a full subcomplex of $LK(f(A))$. The latter condition means that every simplex of $LK(f(A))$ whose vertices are images under f_A of vertices of $LK(A)$ is itself the image under f_A of a simplex of $LK(A)$.

Lemma 2.1. [CW, Theorem 1.2] *If $f : X \rightarrow Y$ is a local isometry of nonpositively curved finite-dimensional cubed complexes, then the induced group homomorphism $f_* : \pi_1(X, A) \rightarrow \pi_1(Y, f(A))$ is injective for all $A \in X$.*

3. GLIDES

3.1. Gliding systems. For a subset \mathcal{G} of a group G , we set $\overline{\mathcal{G}} = \{s^{-1} \mid s \in \mathcal{G}\} \subset G$. A *gliding system* in a group G is a pair of sets

$$\mathcal{G} \subset G \setminus \{1\}, \quad \mathcal{I} \subset \mathcal{G} \times \mathcal{G}$$

such that $\overline{\mathcal{G}} = \mathcal{G}$ and for any $s, t \in \mathcal{G}$ with $(s, t) \in \mathcal{I}$, we have $(s^{-1}, t), (t, s) \in \mathcal{I}$, and $st = ts \neq 1$. The elements of \mathcal{G} are called *glides*. The inverse of a glide is a glide while the unit $1 \in G$ is never a glide. The set \mathcal{I} determines a symmetric

relation on \mathcal{G} called *independence*. Thus, two glides s, t are independent if and only if $(s, t) \in \mathcal{I}$. For any independent glides s, t , we have $st = ts$, $s \neq t^{\pm 1}$, and $(s^\varepsilon, t^\mu), (t^\mu, s^\varepsilon) \in \mathcal{I}$ for all $\varepsilon, \mu = \pm 1$. In particular, a glide is never independent from itself or its inverse. We do not require the glides to generate G as a group.

Let $(\mathcal{G}, \mathcal{I})$ be a gliding system in a group G . For $s \in \mathcal{G}$ and $A \in G$, we say that $sA \in G$ is obtained from A by *(left) gliding along s* . One can similarly consider right glidings but we do not need them.

A subset S of \mathcal{G} is *pre-cubic* if it is finite and $(s, t) \in \mathcal{I}$ for any distinct $s, t \in S$. Since independent glides commute, such a set S determines an element $[S] = \prod_{s \in S} s$ of G . In particular, the empty set $\emptyset \subset \mathcal{G}$ is pre-cubic and $[\emptyset] = 1$.

A set $S \subset \mathcal{G}$ is *cubic* if it is pre-cubic and for any distinct subsets T_1, T_2 of S , we have $[T_1] \neq [T_2]$. In particular, $[T] \neq [\emptyset] = 1$ for any non-empty $T \subset S$. Examples of cubic sets of glides: \emptyset ; a set consisting of a single glide; a set consisting of two independent glides. It is clear that any subset of a cubic set of glides is cubic.

Lemma 3.1. *For each subset T of a (pre-)cubic set of glides $S \subset G$, the set of glides $S_T = (S \setminus T) \cup \overline{T}$ is (pre-)cubic where $\overline{T} = \{t^{-1} \mid t \in T\} \subset G$.*

Proof. That S_T is finite and consists of pairwise independent glides follows from the definitions. We need to prove that if S is cubic, then so is S_T . It suffices to show that $[T_1] = [T_2] \implies T_1 = T_2$ for any subsets T_1, T_2 of S_T . For $i = 1, 2$, put $U_i = (S \setminus T) \cap T_i$ and $V_i = T \cap \overline{T}_i$. Clearly, $\overline{V}_i = \overline{T} \cap T_i$ and $U_i \cup \overline{V}_i = T_i$. The independence of the elements of S implies that the sets $S \setminus T$ and \overline{T} are disjoint. Therefore their subsets U_i and \overline{V}_i are disjoint and

$$[T_i] = [U_i \cup \overline{V}_i] = [U_i][\overline{V}_i] = [U_i][V_i]^{-1}.$$

The assumption $[T_1] = [T_2]$ implies that $[U_1][V_1]^{-1} = [U_2][V_2]^{-1}$. The sets V_1, V_2 are subsets of $T \subset S$ and therefore the elements $[V_1], [V_2]$ of G commute. Hence $[U_1][V_2] = [U_2][V_1]$. The equalities $U_1 \cap V_2 = U_2 \cap V_1 = \emptyset$ imply that

$$[U_1 \cup V_2] = [U_1][V_2] = [U_2][V_1] = [U_2 \cup V_1].$$

Since $U_1 \cup V_2$ and $U_2 \cup V_1$ are subsets of the cubic set S , they must be equal. So,

$$U_1 = (U_1 \cup V_2) \cap (S \setminus T) = (U_2 \cup V_1) \cap (S \setminus T) = U_2$$

and

$$V_1 = (U_2 \cup V_1) \cap T = (U_1 \cup V_2) \cap T = V_2.$$

We conclude that $T_1 = U_1 \cup \overline{V}_1 = U_2 \cup \overline{V}_2 = T_2$. \square

3.2. Examples. 1. For any group G , the following pair is a gliding system: $\mathcal{G} = G \setminus \{1\}$ and \mathcal{I} is the set of all pairs $(s, t) \in \mathcal{G} \times \mathcal{G}$ such that $s \neq t^{\pm 1}$ and $st = ts$.

2. For a group G and a set $\mathcal{G} \subset G \setminus \{1\}$ such that $\overline{\mathcal{G}} = \mathcal{G}$, the pair $(\mathcal{G}, \mathcal{I} = \emptyset)$ is a gliding system. The cubic subsets of \mathcal{G} are the empty set and the 1-element subsets.

3. Let G be a free abelian group with free commuting generators $\{g_i\}_i$. Then

$$\mathcal{G} = \{g_i^{\pm 1}\}_i, \quad \mathcal{I} = \{(g_i^\varepsilon, g_j^\mu) \mid \varepsilon = \pm 1, \mu = \pm 1, i \neq j\}$$

is a gliding system in G . A cubic subset of \mathcal{G} consists of a finite number of g_i and a finite number of g_j^{-1} with $i \neq j$.

4. A generalization of the previous example is provided by the theory of right-angled Artin groups (see [Ch] for an exposition). A right-angled Artin group is a group allowing a presentation by generators and relations in which all relators are

commutators of the generators. Any graph Γ with the set of vertices V determines a right-angled Artin group $G = G(\Gamma)$ with generators $\{g_s\}_{s \in V}$ and relations $g_s g_t = g_t g_s$ whenever $s, t \in V$ are connected by an edge in Γ (we write then $s \leftrightarrow t$). Abelianizing G we obtain that $g_s \neq g_t^{\pm 1}$ for $s \neq t$. The pair

$$\mathcal{G} = \{g_s^{\pm 1}\}_{s \in V}, \quad \mathcal{I} = \{(g_s^\varepsilon, g_t^\mu) \mid \varepsilon, \mu = \pm 1, s, t \in V, s \neq t, s \leftrightarrow t\}$$

is a gliding system in G .

5. Let E be a set and $G = 2^E$ be the power set of E consisting of all subsets of E . We define multiplication in G by $AB = (A \cup B) \setminus (A \cap B)$ for $A, B \subset E$. This turns G into an abelian group with unit $1 = \emptyset$, the *power group of E* . Clearly, $A^{-1} = A$ for all $A \in G$. Pick any set $\mathcal{G} \subset G \setminus \{1\}$ and declare elements of \mathcal{G} independent when they are disjoint as subsets of E . This gives a gliding system in G . A cubic subset of \mathcal{G} is just a finite collection of pairwise disjoint non-empty subsets of E .

6. Let E be a set, H be a multiplicative group, and $G = H^E$ be the group of all maps $E \rightarrow H$ with pointwise multiplication. The *support* of a map $f : E \rightarrow H$ is the set $\text{supp}(f) = f^{-1}(H \setminus \{1\}) \subset E$. Pick a set $\mathcal{G} \subset G \setminus \{1\}$ invariant under inversion and declare elements of \mathcal{G} independent if their supports are disjoint. This gives a gliding system in G . When H is a cyclic group of order 2, we recover Example 5 via the group isomorphism $H^E \simeq 2^E$ carrying a map $E \rightarrow H$ to its support.

3.3. Regular gliding systems. A gliding system is *regular* if all pre-cubic sets of glides in this system are cubic. We will be mainly interested in regular gliding systems. The gliding system in Example 3.2.1 may be non-regular while those in Examples 3.2.2–6 are regular. The regularity in Examples 3.2.5 and 3.2.6 is a consequence of the following lemma.

Lemma 3.2. *Let E be a set and H be a group. Consider a gliding system in the group $G = H^E$ such that the supports of any two independent glides are disjoint (as subsets of E). Then this gliding system is regular.*

Proof. Consider an arbitrary pre-cubic set of glides $S \subset G$. Any subset T of S is also pre-cubic. Since the supports of independent glides are disjoint,

$$\text{supp}([T]) = \coprod_{s \in T} \text{supp}(s) \subset E.$$

Consider two distinct subsets T_1, T_2 of S . Assume for concreteness that $T_1 \setminus T_2 \neq \emptyset$. Pick $t \in T_1 \setminus T_2$. Then $\text{supp}(t) \subset \text{supp}([T_1])$ and $\text{supp}(t) \cap \text{supp}([T_2]) = \emptyset$. Since t is a glide, $\text{supp}(t) \neq \emptyset$. Hence $[T_1] \neq [T_2]$. This proves that the set S is cubic. \square

4. GLIDE COMPLEXES AND GLIDE GROUPS

In this section, G is a group equipped with a gliding system. We define a cubed complex X_G , the *glide complex*, and study certain cubed subcomplexes of X_G .

4.1. The glide complex. A *based cube* in G is a pair $(A \in G, \text{a cubic set of glides } S \subset G)$. The integer $k = \text{card}(S) \geq 0$ is the *dimension* of the based cube (A, S) . It follows from the definition of a cubic set of glides that the set $\{[T]A\}_{T \subset S} \subset G$ has 2^k elements; these elements are called the *vertices* of the based cube (A, S) .

Two based cubes (A, S) and (A', S') are *equivalent* if there is a set $T \subset S$ such that $A' = [T]A$ and $S' = S_T$ where S_T is defined in Lemma 3.1. This is indeed an equivalence relation on the set of based cubes. Each k -dimensional based cube is equivalent to 2^k based cubes (including itself). The equivalence classes of k -dimensional based cubes are called *k -dimensional cubes* or *k -cubes* in G . Since

equivalent based cubes have the same vertices, we may speak of the vertices of a k -cube. The 0-cubes in G are just the elements of G .

A cube Q in G is a *face* of a cube Q' in G if Q, Q' may be represented by based cubes $(A, S), (A', S')$, respectively, such that $A = A'$ and $S \subset S'$. Note that in the role of A one may take an arbitrary vertex of Q .

The glide complex X_G is the cubed complex obtained by taking a copy of I^k for each k -cube in G with $k \geq 0$ and gluing these copies via identifications determined by inclusions of cubes into bigger cubes as their faces. Here is a more precise definition. A point of X_G is represented by a triple $(A, S, x \in I^S)$ where (A, S) is a based cube in G and I^S is the set of all maps $S \rightarrow I$ viewed as the product of copies of I numerated by elements of S and provided with the product topology. For fixed (A, S) , the triples (A, S, x) form the geometric cube I^S . We take a disjoint union of these cubes over all (A, S) and factorize it by the equivalence relation generated by the relation $(A, S, x) \sim (A', S', x')$ when

$$A = A', \quad S \subset S', \quad x = x'|_S, \quad x'(S' \setminus S) = 0$$

or there is a set $T \subset S$ such that

$$A' = [T]A, \quad S' = S_T, \quad x'|_{S' \setminus T} = x|_{S \setminus T} \text{ and } x(t) + x'(t^{-1}) = 1 \text{ for all } t \in T.$$

The quotient space X_G is a cubed space in the obvious way.

Lemma 4.1. *All cubes in G are embedded in X_G .*

Proof. This follows from the fact that different faces of a k -cube have different sets of vertices and therefore are never glued to each other under our identifications. \square

4.2. Subcomplexes of X_G . Any set $\mathcal{D} \subset G$ determines a cubed complex $X_{\mathcal{D}} \subset X_G$ formed by the cubes in G whose all vertices belong to \mathcal{D} . Such cubes are called *cubes in \mathcal{D}* , and $X_{\mathcal{D}}$ is called the *glide complex* of \mathcal{D} . For $A \in \mathcal{D}$, the fundamental group $\pi_1(X_{\mathcal{D}}, A)$ is called the *glide group of \mathcal{D} at A* . The elements of \mathcal{D} related by glidings in \mathcal{D} belong to the same component of $X_{\mathcal{D}}$ and give rise to isomorphic glide groups.

Lemma 4.2. *For any set $\mathcal{D} \subset G$, the cubed complex $X_{\mathcal{D}}$ is simple in the sense of Section 2.1.*

Proof. A neighborhood of $A \in G$ in X_G can be obtained by taking all triples (A, S, x) , where S is a cubic set of glides and $x(S) \subset [0, 1/2)$, and identifying two such triples $(A, S_1, x_1), (A, S_2, x_2)$ whenever $x_1 = x_2$ on $S_1 \cap S_2$ and $x_1(S_1 \setminus S_2) = x_2(S_2 \setminus S_1) = 0$. Therefore, the link $LK_G(A)$ of A in X_G is the simplicial complex whose vertices are the glides and whose simplices are the cubic sets of glides. The link $LK_{\mathcal{D}}(A)$ of $A \in \mathcal{D}$ in $X_{\mathcal{D}}$ is the subcomplex of $LK_G(A)$ formed by the glides $s \in G$ such that $sA \in \mathcal{D}$ and the cubic sets $S \subset G$ such that $[T]A \in \mathcal{D}$ for all $T \subset S$. \square

For $A \in \mathcal{D}$, we now reformulate the flag condition on the link $LK_{\mathcal{D}}(A)$ of A in $X_{\mathcal{D}}$ in terms of glides. Observe that finite sets of pairwise adjacent vertices in $LK_{\mathcal{D}}(A)$ bijectively correspond to pre-cubic sets of glides $S \subset G$ such that

$$(*) \quad sA \in \mathcal{D} \text{ for all } s \in S \text{ and } stA \in \mathcal{D} \text{ for all distinct } s, t \in S.$$

Lemma 4.3. *The link $LK_{\mathcal{D}}(A)$ of $A \in \mathcal{D}$ in $X_{\mathcal{D}}$ is a flag complex if and only if any pre-cubic set of glides $S \subset G$ satisfying $(*)$ is cubic and $[S]A \in \mathcal{D}$.*

This lemma follows directly from the definitions. One should use the obvious fact that if a pre-cubic set of glides satisfies $(*)$ then so do all its subsets.

We now formulate combinatorial conditions on a set $\mathcal{D} \subset G$ necessary and sufficient for $X_{\mathcal{D}}$ to be nonpositively curved in the sense of Section 2.1. We say that \mathcal{D} is *regular* if for every $A \in \mathcal{D}$, all pre-cubic sets of glides $S \subset G$ satisfying $(*)$ are cubic. We say that \mathcal{D} satisfies the *cube condition* if for any $A \in \mathcal{D}$ and any pairwise independent glides $s_1, s_2, s_3 \in G$ such that $s_1A, s_2A, s_3A, s_1s_2A, s_1s_3A, s_2s_3A \in \mathcal{D}$, we necessarily have $s_1s_2s_3A \in \mathcal{D}$. This condition may be reformulated by saying that if seven vertices of a 3-cube in G belong to \mathcal{D} , then so does the eighth vertex.

Theorem 4.4. *The cubed complex $X_{\mathcal{D}}$ of a set $\mathcal{D} \subset G$ is nonpositively curved if and only if \mathcal{D} is regular and satisfies the cube condition.*

Proof. Lemmas 4.2 and 4.3 imply that $X_{\mathcal{D}}$ is nonpositively curved if and only if for all $A \in \mathcal{D}$, each pre-cubic set of glides $S \subset G$ satisfying $(*)$ is cubic and $[S]A \in \mathcal{D}$. We must only show that the inclusion $[S]A \in \mathcal{D}$ can be replaced with the cube condition. One direction is obvious: if $A, s_1, s_2, s_3 \in G$ satisfy the assumptions of the cube condition, then the set $S = \{s_1, s_2, s_3\}$ satisfies $(*)$ and so $s_1s_2s_3A = [S]A \in \mathcal{D}$. Conversely, suppose that \mathcal{D} is regular and meets the cube condition. We must show that $[S]A \in \mathcal{D}$ for any $A \in \mathcal{D}$ and any pre-cubic set of glides $S \subset G$ satisfying $(*)$. We proceed by induction on $k = \text{card}(S)$. For $k = 0$, the claim follows from the inclusion $A \in \mathcal{D}$. For $k = 1, 2$, the claim follows from $(*)$. If $k \geq 3$, then the induction assumption guarantees that $[T]A \in \mathcal{D}$ for any proper subset $T \subset S$. If $S = \{s_1, \dots, s_k\}$, then applying the cube condition to s_1, s_2, s_3 and the element $s_4 \cdots s_kA$ of \mathcal{D} , we obtain that $[S]A \in \mathcal{D}$. \square

By Section 2.1, if $X_{\mathcal{D}}$ is nonpositively curved, then the universal covering of every finite-dimensional component of $X_{\mathcal{D}}$ is a CAT(0)-space. The component itself is then an Eilenberg-MacLane space of type $K(\pi, 1)$ where π is the corresponding glide group. Note that $\dim X_{\mathcal{D}}$ is the maximal dimension of a cube in \mathcal{D} . In particular, if \mathcal{D} is a finite set, then $X_{\mathcal{D}}$ is a finite-dimensional complex.

If the gliding system in G is regular, then all subsets of G are regular. As a consequence, we obtain the following corollary.

Corollary 4.5. *For a group G with a regular gliding system, the glide complex $X_{\mathcal{D}}$ of a set $\mathcal{D} \subset G$ is nonpositively curved if and only if \mathcal{D} satisfies the cube condition. In particular, the glide complex X_G of $\mathcal{D} = G$ is nonpositively curved.*

Corollary 4.5 applies, in particular, to the regular gliding systems described in Examples 3.2.2–6.

4.3. Remarks. We make a few miscellaneous remarks on the glide complexes.

1. The group G acts on X_G on the right by $(A, S, x)g = (Ag, S, x)$ for $g \in G$. This action preserves the cubed structure and is free and transitive on $X_G^{(0)} = G$.
2. In Example 3.2.2, X_G is the graph with the set of vertices G , two vertices $A, B \in G$ being connected by a (single) edge if and only if $AB^{-1} \in \mathcal{G}$. All subsets of G satisfy the cube condition because G has no independent glides. The corresponding glide groups are free.
3. In Example 3.2.3, if the rank, n , of G is finite, then $X_G = \mathbb{R}^n$ with the standard action of \mathbb{Z}^n .
4. In Example 3.2.4, X_G is simply-connected, the action of G on X_G is free, and the projection $X_G \rightarrow X_G/G$ is the universal covering of X_G/G . Composing the cells

of X_G with this projection we turn X_G/G into a cubed complex called the Salvetti complex, see [Ch]. This complex has only one 0-cell; its link is isomorphic to the link of any vertex of X_G and is a flag complex. This recovers the well known fact that the Salvetti complex is nonpositively curved. Clearly, $\dim X_G = \dim(X_G/G)$ is the maximal number of vertices of a complete subgraph of the graph Γ .

5. In Examples 3.2.5 and 3.2.6, if E is finite, then X_G is finite dimensional.

5. HOMOMORPHISMS OF GLIDE GROUPS

We discuss two families of homomorphisms of glide groups: the inclusion homomorphisms and the typing homomorphisms.

5.1. Inclusion homomorphisms. Let G be a group with a gliding system. Consider any sets $\mathcal{E} \subset \mathcal{D} \subset G$ and the associated cubed complexes $X_{\mathcal{E}} \subset X_{\mathcal{D}} \subset X_G$. We say that \mathcal{E} satisfies the *square condition* $\text{rel } \mathcal{D}$, if for any $A \in \mathcal{E}$ and any independent glides $s, t \in G$ such that $sA, tA \in \mathcal{E}$, $stA \in \mathcal{D}$, we necessarily have $stA \in \mathcal{E}$. This condition may be reformulated by saying that if three vertices of a square (a 2-cube) in $X_{\mathcal{D}}$ belong to $X_{\mathcal{E}}$, then so does the fourth vertex. For example, the intersection of \mathcal{D} with any subgroup of G satisfies the square condition $\text{rel } \mathcal{D}$.

Theorem 5.1. *Let $\mathcal{D} \subset G$ be a regular set satisfying the cube condition and such that $\dim X_{\mathcal{D}} < \infty$. Let \mathcal{E} be a subset of \mathcal{D} satisfying the square condition $\text{rel } \mathcal{D}$. Then the inclusion homomorphism $\pi_1(X_{\mathcal{E}}, A) \rightarrow \pi_1(X_{\mathcal{D}}, A)$ is injective for all $A \in \mathcal{E}$.*

Proof. Since \mathcal{D} is regular, so is $\mathcal{E} \subset \mathcal{D}$. The cube condition on \mathcal{D} and the square condition $\text{rel } \mathcal{D}$ on \mathcal{E} imply that \mathcal{E} satisfies the cube condition. By Theorem 4.4, $X_{\mathcal{D}}$ and $X_{\mathcal{E}}$ are nonpositively curved. By assumption, the cubed complex $X_{\mathcal{D}}$ is finite-dimensional and so is its subcomplex $X_{\mathcal{E}}$. We claim that the inclusion $X_{\mathcal{E}} \hookrightarrow X_{\mathcal{D}}$ is a local isometry. Together with Lemma 2.1 this will imply the theorem.

To prove our claim, pick any $A \in \mathcal{E}$ and consider the simplicial complexes $L' = LK(A, X_{\mathcal{E}})$, $L = LK(A, X_{\mathcal{D}})$. The inclusion $X_{\mathcal{E}} \hookrightarrow X_{\mathcal{D}}$ induces an embedding $L' \hookrightarrow L$, and we need only to verify that the image of L' is a full subcomplex of L . Since L' is a flag simplicial complex, it suffices to verify that any vertices of L' adjacent in L are adjacent in L' . This follows from the square condition on \mathcal{E} . \square

The set $\mathcal{D} \subset G$ satisfies the *square condition* if it satisfies the square condition $\text{rel } G$. In other words, \mathcal{D} satisfies the square condition if for any $A \in \mathcal{D}$ and any independent glides $s, t \in G$ with $sA, tA \in \mathcal{D}$, we necessarily have $stA \in \mathcal{D}$.

Corollary 5.2. *If the gliding system in G is regular and $\dim X_G < \infty$, then for every set $\mathcal{D} \subset G$ satisfying the square condition and every $A \in \mathcal{D}$, the inclusion homomorphism $\pi_1(X_{\mathcal{D}}, A) \rightarrow \pi_1(X_G, A)$ is injective.*

5.2. Typing homomorphisms. A group G carrying a gliding system $(\mathcal{G}, \mathcal{I})$ determines a right-angled Artin group $\mathcal{A} = \mathcal{A}(G)$ with generators $\{g_s\}_{s \in \mathcal{G}}$ and relations $g_s g_t = g_t g_s$ where (s, t) runs over \mathcal{I} . We now relate \mathcal{A} to the glide groups.

First, we introduce a notion of an orientation on a set $\mathcal{D} \subset G$. An *orientation* on \mathcal{D} is a choice of direction on each 1-cell of the glide complex $X_{\mathcal{D}}$ such that the opposite sides of any 2-cell of $X_{\mathcal{D}}$ (a square) point towards each other on the boundary of the square. In other words, for any based square $(A, \{s, t\})$ with $A, sA, tA, stA \in \mathcal{D}$, the 1-cells connecting A to sA and tA to stA are either both directed towards sA, stA or both directed towards A, tA (and similarly with s, t

exchanged). A set $\mathcal{D} \subset G$ is *orientable* if it allows an orientation and is *oriented* if it has a distinguished orientation. An orientation of \mathcal{D} induces an orientation of any subset $\mathcal{E} \subset \mathcal{D}$ via the inclusion $X_{\mathcal{E}} \subset X_{\mathcal{D}}$. Therefore, all subsets of an orientable set are orientable. These definitions apply, in particular, to $\mathcal{D} = G$. Examples of oriented sets will be given in Section 6.2.

Consider an oriented set $\mathcal{D} \subset G$. Each 1-cell e of $X_{\mathcal{D}}$ is oriented and so leads from a vertex $A \in \mathcal{D}$ to a vertex $B \in \mathcal{D}$. We set $|e| = BA^{-1} \in G$. It follows from the definition of $X_{\mathcal{D}}$ that $|e|$ is a glide in G . Consider now a path α in the 1-skeleton of $X_{\mathcal{D}}$ formed by $n \geq 0$ consecutive 1-cells e_1, \dots, e_n . The path α determines an orientation of e_1, \dots, e_n so that the terminal endpoint of e_k is the initial endpoint of e_{k+1} for $k = 1, \dots, n-1$. This orientation of e_k may coincide or not with that given by the orientation of \mathcal{D} . We set $\nu_k = +1$ or $\nu_k = -1$, respectively. Set

$$(5.2.1) \quad \mu(\alpha) = g_{|e_1|}^{\nu_1} g_{|e_2|}^{\nu_2} \cdots g_{|e_n|}^{\nu_n} \in \mathcal{A} = \mathcal{A}(G).$$

It is clear that $\mu(\alpha)$ is preserved under inserting in the sequence e_1, \dots, e_n two opposite 1-cells or four 1-cells forming the boundary of a 2-cell. Therefore $\mu(\alpha)$ is preserved under homotopies of α in $X_{\mathcal{D}}$ relative to the endpoints. Applying μ to loops based at $A \in \mathcal{D}$, we obtain a homomorphism $\mu_A : \pi_1(X_{\mathcal{D}}, A) \rightarrow \mathcal{A}$. Following the terminology of [HaW], we call μ_A the *typing homomorphism*.

Theorem 5.3. *If there is an upper bound on the number of pairwise independent glides in G , then for any oriented regular set $\mathcal{D} \subset G$ satisfying the square condition and for any $A \in \mathcal{D}$, the typing homomorphism $\mu_A : \pi_1(X_{\mathcal{D}}, A) \rightarrow \mathcal{A}$ is injective.*

Proof. By Example 3.2.4, the group \mathcal{A} carries a gliding system with glides $\{g_s^{\pm 1}\}_{s \in \mathcal{G}}$. Two glides $g_s^{\pm 1}, g_t^{\pm 1} \in \mathcal{A}$ are independent if and only if $s \neq t$ and $(s, t) \in \mathcal{I}$. Consider the associated cubed complex $X = X_{\mathcal{A}}$ and the Salvetti complex $Y = X/\mathcal{A}$. Recall that $\mathcal{A} = \pi_1(Y, *)$ where $*$ is the unique 0-cell of Y . By Section 4.3, both X and Y are nonpositively curved. The assumptions of the theorem imply that the cubed complex $X_{\mathcal{D}}$ is nonpositively curved, and the spaces $X, Y, X_{\mathcal{D}}$ are finite-dimensional. We claim that the homomorphism $\mu_A : \pi_1(X_{\mathcal{D}}, A) \rightarrow \mathcal{A}$ is induced by a local isometry $X_{\mathcal{D}} \rightarrow Y$. By Lemma 2.1, this will imply the theorem.

Since the gliding system in \mathcal{A} is regular, the points of X are represented by triples $(A \in \mathcal{A}, \mathcal{S}, x \in I^{\mathcal{S}})$ where \mathcal{S} is a pre-cubic set of glides in \mathcal{A} that is $\mathcal{S} = \{g_s^{\varepsilon_s}\}_s$ where s runs over a finite set of independent glides in G and $\varepsilon_s \in \{\pm 1\}$. The space X is obtained by factorizing the set of such triples by the equivalence relation defined in Section 4.1. The space Y is obtained from X by forgetting the first term, A , of the triple. A point of Y is represented by a pair (a pre-cubic set of glides $\mathcal{S} \subset \mathcal{A}, x \in I^{\mathcal{S}}$). The space Y is obtained by factorizing the set of such pairs by the equivalence relation generated by the following relation: $(\mathcal{S}, x) \sim (\mathcal{S}', x')$ when $\mathcal{S} \subset \mathcal{S}', x = x'|_{\mathcal{S}}, x'(\mathcal{S}' \setminus \mathcal{S}) = 0$ or there is $T \subset \mathcal{S}$ such that $\mathcal{S}' = \mathcal{S}_T, x' = x$ on $\mathcal{S} \setminus T$, and $x'(t^{-1}) = 1 - x(t)$ for all $t \in T$.

We now construct a cubical map $f : X_{\mathcal{D}} \rightarrow Y$ carrying each 1-cell e of $X_{\mathcal{D}}$ onto the 1-cell of Y determined by $g_{|e|}$ where $|e| \in G$ is the glide determined by the distinguished orientation of e . Here is a precise definition of f . A point $a \in X_{\mathcal{D}}$ is represented by a triple $(A \in \mathcal{D}, \mathcal{S}, x \in I^{\mathcal{S}})$ where \mathcal{S} is a cubic set of glides in G such that $[T]A \in \mathcal{D}$ for all $T \subset \mathcal{S}$. For $s \in \mathcal{S}$, set $|s| = |e_s|$ where e_s is the 1-cell of $X_{\mathcal{D}}$ connecting A and sA . By definition, $|s| = s^{\varepsilon_s}$ where $\varepsilon_s = +1$ if e_s is oriented towards sA and $\varepsilon_s = -1$ otherwise. Let $f(a) \in Y$ be represented by the pair

($\mathcal{S} = \{g_{|s|}^{\varepsilon_s}\}_{s \in S}, y \in I^{\mathcal{S}}$) where $y(g_{|s|}^{\varepsilon_s}) = x(s)$ for all $s \in S$. This yields a well-defined cubical map $f : X_{\mathcal{D}} \rightarrow Y$ inducing μ_A in π_1 .

It remains to show that f is a local isometry. The link $L = LK_{\mathcal{D}}(A)$ of $A \in \mathcal{D}$ in $X_{\mathcal{D}}$ has a vertex v_s for every glide $s \in G$ such that $sA \in \mathcal{D}$. A set of vertices $\{v_s\}_s$ spans a simplex in L whenever s runs over a cubic set of glides in G (cf. the proof of Lemma 4.2; here we use the square condition on \mathcal{D}). The link, K , of $*$ in Y has two vertices w_s^+ and w_s^- for every glide $s \in G$. A set of vertices $\{w_s^{\pm}\}_s$ spans a simplex in K whenever s runs over a pre-cubic set of glides. The map $f_A : L \rightarrow K$ induced by f carries v_s to $w_{|s|}^{\varepsilon_s}$. This map is an embedding since we can recover s from $|s|$ and ε_s . The square condition on \mathcal{D} implies that if the images of two vertices of L under f_A are adjacent in K , then the vertices themselves are adjacent in L . Since L is a flag simplicial complex, $f_A(L)$ is a full subcomplex of K . \square

Corollary 5.4. *If the set of glides in G is finite and an orientable regular set $\mathcal{D} \subset G$ satisfies the square condition, then for all $A \in \mathcal{D}$, the glide group $\pi_1(X_{\mathcal{D}}, A)$ is biorderable, residually nilpotent, residually finite, and embeds in $SL_n(\mathbb{Z})$ for some n .*

Proof. Finitely generated right-angled Artin groups have all the properties listed in this corollary, see [DuT], [DaJ], [CW], [HsW]. These properties are hereditary and so are shared by all subgroups of finitely generated right-angled Artin groups. Combining with Theorem 5.3 we obtain the desired result. \square

Corollary 5.5. *If the gliding system in G is regular, the number of pairwise independent glides in G is bounded from above, and G is oriented in the sense of Section 5.2, then $\mu_A : \pi_1(X_G, A) \rightarrow \mathcal{A}$ is an injection for all $A \in G$.*

Proof. This follows from Theorem 5.3 because the square condition on G is void. \square

6. THE DIMER COMPLEX AND THE DIMER GROUP

6.1. Cycles in graphs. By a *graph* we mean a non-empty 1-dimensional CW-complex without isolated 0-cells (i.e., 0-cells not incident to any 1-cells) and without loops (i.e., 1-cells with equal endpoints). The 0-cells and 1-cells of a graph are called *vertices* and *edges*, respectively. We allow multiple edges with the same endpoints. A *subgraph* of a graph Γ is a graph Γ' embedded in Γ such that all vertices/edges of Γ' are also vertices/edges of Γ .

Given a set s of edges of a graph Γ , we denote by ∂s the set of vertices of Γ adjacent to at least one edge in s . Such vertices of Γ are called *vertices of s* . The set s is *cyclic* if it is finite and each vertex of s is incident to precisely two edges in s . The vertices of a cyclic set s together with the edges in s form a subgraph of Γ denoted \underline{s} and homeomorphic to a disjoint union of a finite number of circles. A cyclic set of edges s is a *cycle* if \underline{s} is a single circle. A cycle is *even* (respectively, *odd*) if it includes an even (respectively, *odd*) number of edges of Γ . Any even cycle has a unique partition into two subsets called the *halves* such that the edges belonging to the same half have no common vertices.

6.2. The even-cycle gliding system. Let Γ be a graph with the set of edges E , and let $G = G(\Gamma) = 2^E$ be the power group of E . Two sets $s, t \subset E$ are *independent* if the edges belonging to s have no common vertices with the edges belonging to t . Such sets s, t are necessarily disjoint.

Lemma 6.1. *The even cycles in Γ in the role of glides together with the independence relation above form a regular gliding system in G .*

All axioms of a gliding system are straightforward. The regularity follows from Lemma 3.2. We call the resulting gliding system in G the *even-cycle* gliding system. By Section 4.1, it determines a cubed complex X_G with 0-skeleton G . By Corollary 4.5, X_G is nonpositively curved.

We show how to orient G in the sense of Section 5.2. Pick an element $e_s \in s$ in every even cycle $s \subset E$. A 1-cell of X_G relates two 0-cells $A, B \subset E$ such that $AB = (A \setminus B) \cup (B \setminus A) \subset E$ is an even cycle. Then $e_{AB} \in AB$ belongs either to A or to B . We orient this 1-cell towards the 0-cell containing e_{AB} . It is easy to see that this procedure defines an orientation on G . By Section 5.2, the latter determines a typing homomorphism $\pi_1(X_G, A) \rightarrow \mathcal{A}(G)$ for $A \in G$.

All cycles in Γ (even and odd) with the independence relation above also form a regular gliding system and yield a nonpositively curved cubed complex. Some of our results extend to this gliding system but we will not study it.

6.3. Perfect matchings. Let Γ be a graph with the set of edges E . We provide the power group $G = 2^E$ with the even-cycle gliding system. A *perfect matching*, or a *dimer covering*, on Γ is a subset of E such that every vertex of Γ is incident to exactly one edge in this subset. Let $\mathcal{D} = \mathcal{D}(\Gamma) \subset G$ be the set (possibly, empty) of all perfect matchings on Γ . By Section 4.2, this set determines a cubed complex $X_{\mathcal{D}} = X_{\mathcal{D}}(\Gamma) \subset X_G$ with 0-skeleton \mathcal{D} . We call $X_{\mathcal{D}}$ the *dimer complex* of Γ . By definition, two perfect matchings $A, B \subset E$ are connected by an edge in $X_{\mathcal{D}}$ if and only if $AB \subset E$ is an even cycle. Note that if AB is a cycle then this cycle is even with halves $A \setminus B$ and $B \setminus A$.

Lemma 6.2. *The set of perfect matchings $\mathcal{D} \subset G$ satisfies the cube condition of Section 4.2 and the square condition of Section 5.1.*

Proof. The cube condition follows from the square condition. The latter says that for any $A \in \mathcal{D}$ and any independent even cycles s, t in Γ such that $sA, tA \in \mathcal{D}$, we must have $stA \in \mathcal{D}$. The inclusions $A, sA \in \mathcal{D}$ imply that $s \cap A$ and $s \setminus A$ are the halves of s . The inclusions $A, tA \in \mathcal{D}$ imply that $t \cap A$ and $t \setminus A$ are the halves of t . The independence of s, t ensures that s, t are disjoint and incident to disjoint sets of vertices. The set $stA \subset E$ is obtained from A through simultaneous replacement of the half $s \cap A$ of s and the half $t \cap A$ of t with the complementary halves. It is clear that stA is a dimer covering. \square

Theorem 6.3. *The dimer complex $X_{\mathcal{D}}$ is nonpositively curved.*

This theorem follows from Lemma 6.2 and Corollary 4.5.

We next determine when two perfect matchings A, B in Γ belong to the same connected component of $X_{\mathcal{D}}$. We say that A, B are *congruent* if the set $AB = (A \setminus B) \cup (B \setminus A)$ is finite.

Lemma 6.4. *Two perfect matchings in Γ belong to the same connected component of $X_{\mathcal{D}}$ if and only if they are congruent.*

Proof. Pick any $A, B \in \mathcal{D} \subset X_{\mathcal{D}}$. If A, B belong to the same connected component of $X_{\mathcal{D}}$, then A may be connected to B by a sequence of edges in $X_{\mathcal{D}}$. In other words, A can be obtained from B by a finite sequence of glidings along even cycles. Each such gliding removes a finite set of edges from the matching and adds another

finite set of edges. Hence, $AB = (A \setminus B) \cup (B \setminus A)$ is a finite set. Conversely, suppose that the set AB is finite. The definition of a perfect matching implies that AB is cyclic. Any finite cyclic set of edges in Γ splits uniquely as a union of independent cycles. Let $AB = \cup_{i=1}^n s_i$ be such a splitting with $n \geq 0$. All the cycles s_1, \dots, s_n are even: their halves are their intersections with $A \setminus B$ and $B \setminus A$. Then

$$A, s_1 A, s_2 s_1 A, \dots, s_n \cdots s_2 s_1 A = B$$

is a sequence of perfect matchings in which every term is obtain from the previous one by gliding along an even cycle. Hence A is connected to B by a sequence of edges in $X_{\mathcal{D}}$. \square

For a perfect matching A of Γ , the group $\pi_1(X_{\mathcal{D}}, A)$ is called the *dimer group* of Γ at A or, shorter, the *dimer group* of A . Lemma 6.4 implies that the dimer groups of congruent perfect matchings are isomorphic. We will see in Section 7 that the isomorphism in question may be chosen in a canonical way. More precisely, for any congruent perfect matchings A, B in Γ we will define a canonical isomorphism $i_{A,B} : \pi_1(X_{\mathcal{D}}, A) \rightarrow \pi_1(X_{\mathcal{D}}, B)$ and for any $A, B, C \in \mathcal{D}$ we will define an element \overline{ABCA} of $\pi_1(X_{\mathcal{D}}, A)$ such that

- (i) $i_{A,A} = \text{id}$ and $i_{B,A} = i_{A,B}^{-1}$ for any $A, B \in \mathcal{D}$;
- (ii) the automorphism $i_{C,A} i_{B,C} i_{A,B}$ of $\pi_1(X_{\mathcal{D}}, A)$ is the conjugation by \overline{ABCA} for any $A, B, C \in \mathcal{D}$.

6.4. The case of a finite graph. A graph is *finite* if it has a finite number of vertices and edges. Let Γ be a finite graph with the set of edges E and let $G = 2^E$ be the power group of E with the even-cycle gliding system. The cubed complex X_G is a finite CW-space and so is compact and finite dimensional. Since X_G is non-positively curved, it is aspherical. For $A \in G$, the group $\pi_1(X_G, A)$ shares the properties of the fundamental groups of compact finite dimensional nonpositively curved cubed complexes listed in Section 2.1. Also, the right-angled Artin group $\mathcal{A}(G)$ is finitely generated, and the typing homomorphism $\pi_1(X_G, A) \rightarrow \mathcal{A}(G)$ determined by any orientation of G is injective (Corollary 5.5). Therefore $\pi_1(X_G, A)$ shares the properties of finitely generated right-angled Artin groups listed in Corollary 5.4. The same arguments yield similar statements for the dimer complex $X_{\mathcal{D}}$ of Γ and the dimer groups of perfect matchings in Γ . By Corollary 5.2, the inclusion homomorphism $\pi_1(X_{\mathcal{D}}, A) \rightarrow \pi_1(X_G, A)$ is injective for all $A \in \mathcal{D} = \mathcal{D}(\Gamma)$.

Lemma 6.5. *The dimer complex of a finite graph is path connected.*

This lemma is a direct consequence of Lemma 6.4 and the obvious fact that all perfect matchings of a finite graph are congruent. Lemma 6.5 implies that the dimer group $\pi_1(X_{\mathcal{D}}, A)$ does not depend on the choice of $A \in \mathcal{D}$ up to isomorphism. This group, considered up to isomorphism, is called the *dimer group* of Γ and denoted $D(\Gamma)$. By definition, if Γ has no perfect matchings, then $D(\Gamma) = \{1\}$.

6.5. Examples. 1. Let Γ be a triangle (with 3 vertices and 3 edges). The set of glides in $G = G(\Gamma)$ is empty, $X_G = G$ consists of 8 points, $X_{\mathcal{D}} = \mathcal{D} = \emptyset$, and $D(\Gamma) = \{1\}$.

2. Let Γ be a square (with 4 vertices and 4 edges). Then $G = G(\Gamma)$ has one glide, X_G is a disjoint union of 8 closed intervals, $X_{\mathcal{D}}$ is one of them, and $D(\Gamma) = \{1\}$.

3. More generally, let Γ be formed by $n \geq 1$ cyclically connected vertices and $\mathcal{D} = \mathcal{D}(\Gamma)$. If n is odd, then $X_{\mathcal{D}} = \mathcal{D} = \emptyset$. If n is even, then Γ has two perfect matchings, $X_{\mathcal{D}}$ is a segment, and $D(\Gamma) = \{1\}$.

4. Let Γ be formed by 2 vertices and 3 connecting them edges. Then $G = G(\Gamma)$ has 3 glides and X_G is a disjoint union of two complete graphs on 4 vertices. The space $X_{\mathcal{D}}$ is formed by 3 vertices and 3 edges of one of these graphs. Clearly, $D(\Gamma) = \mathbb{Z}$.

5. More generally, for $n \geq 1$, consider the graph Γ^n formed by 2 vertices and n connecting them edges. A perfect matching in Γ^n consists of a single edge, and so, the set $\mathcal{D} = \mathcal{D}(\Gamma^n)$ has n elements. The graph Γ^n has $n(n-1)/2$ cycles, all of length 2 and none of them independent. The complex $X_{\mathcal{D}}$ is a complete graph on n vertices. Hence, $D(\Gamma^n)$ is a free group of rank $(n-1)(n-2)/2$.

6.6. Remarks. 1. If a graph does not have perfect matchings, then one can subdivide some of its edges into two subedges so that the resulting graph has perfect matchings and the theory above applies. Subdivision of edges into three or more subedges is redundant. If a graph Γ' is obtained from a graph Γ by adding an even number of new vertices inside an edge, then there is a canonical bijection $\mathcal{D}(\Gamma) \approx \mathcal{D}(\Gamma')$ which extends to a cubic homeomorphism $X_{\mathcal{D}}(\Gamma) \approx X_{\mathcal{D}}(\Gamma')$.

2. If Γ is a disjoint union of graphs Γ_1, Γ_2 , then $G = G(\Gamma) = G_1 \times G_2$, where $G_i = G(\Gamma_i)$ for $i = 1, 2$, and $X_G = X_{G_1} \times X_{G_2}$. Similarly, $\mathcal{D} = \mathcal{D}(\Gamma) = \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_i = \mathcal{D}(\Gamma_i)$ for $i = 1, 2$, and $X_{\mathcal{D}} = X_{\mathcal{D}_1} \times X_{\mathcal{D}_2}$. If Γ_1, Γ_2 are finite graphs, then $D(\Gamma) = D(\Gamma_1) \times D(\Gamma_2)$.

3. The previous remark and Example 6.5.4 imply that any free abelian group of finite rank is realizable as the dimer group of a finite graph. Other abelian groups cannot be realized as dimer groups of finite graphs because the latter are finitely generated and torsion-free. It would be interesting to find a finite graph whose dimer group is not a product of free groups of rank $n(n-1)/2$ with $n \geq 1$.

4. Consider finite graphs Γ_1, Γ_2 admitting perfect matchings. Let Γ' be obtained from $\Gamma = \Gamma_1 \amalg \Gamma_2$ by adding an edge connecting a vertex of Γ_1 with a vertex of Γ_2 . Then $D(\Gamma') = D(\Gamma)$ and $X_{\mathcal{D}}(\Gamma') = X_{\mathcal{D}}(\Gamma)$. This implies that for any finite graph, there is a connected finite graph with the same dimer group.

7. A PRESENTATION OF THE DIMER GROUP

We give a presentation of the dimer group of a graph by generators and relations. We begin with a general result concerning so-called straight CW-spaces.

7.1. Straight CW-spaces. The cellular structure of a CW-space is commonly used to read from its 2-skeleton a presentation of the fundamental group by generators and relations. We discuss a different method producing a presentation of the fundamental group for some CW-spaces. We call a CW-space X *straight* if all closed cells of X are embedded balls and for any 0-cells $A, B \in X$ the intersection of all closed cells of X containing A, B is a closed cell of X . This intersection is called the *hull* of A, B . Connecting A and B by a path in their hull, we obtain a well-defined homotopy class of paths from A to B in X denoted \overline{AB} . For a finite sequence of 0-cells A_0, A_1, \dots, A_m the product of $\overline{A_0A_1}, \overline{A_1A_2}, \dots, \overline{A_{m-1}A_m}$ is a well-defined homotopy class of paths from A_0 to A_m . It is denoted $\overline{A_0A_1 \dots A_m}$.

Lemma 7.1. *Let X be a straight CW-space, and let $Y = X^{(0)}$ be the 0-skeleton of X . Then X is path-connected and for each $A_0 \in Y$, the group $\pi_1(X, A_0)$ is*

isomorphic to the group with generators $\{x_{A,B}\}_{A,B \in Y}$ subject to the following relations: $x_{A_0,A} = 1$ for all $A \in Y$ and $x_{A,C} = x_{A,B} x_{B,C}$ for each triple $A, B, C \in Y$ such that there is a closed cell of X containing $\{A, B, C\}$.

Proof. The path-connectedness of X is obvious. Let Π be the group defined by the generators and relations in the statement. The relation $x_{A_0,A} x_{A,A} = x_{A_0,A}$ implies that $x_{A,A} = 1$ for all $A \in Y$. Then $x_{A,B} x_{B,A} = x_{A,A} = 1$ for all $A, B \in Y$ so that $x_{B,A} = x_{A,B}^{-1}$. In particular, $x_{A,A_0} = x_{A_0,A}^{-1} = 1$ for all $A \in Y$.

We define a homomorphism $\phi : \Pi \rightarrow \pi_1(X, A_0)$ by $\phi(x_{A,B}) = \overline{A_0 A B A_0}$. This definition is compatible with the relations in Π . Indeed, for $A \in Y$,

$$\phi(x_{A_0,A}) = \overline{A_0 A_0 A A_0} = \overline{A_0 A A_0} = 1.$$

If $A, B, C \in Y$ lie in a closed cell of X , then $\overline{A B C} = \overline{A C}$ and therefore

$$\phi(x_{A,B} x_{B,C}) = \overline{A_0 A B A_0 B C A_0} = \overline{A_0 A B C A_0} = \overline{A_0 A C A_0} = \phi(x_{A,C}).$$

We next define a homomorphism $\psi : \pi_1(X, A_0) \rightarrow \Pi$. Note that if $A, B \in Y$ are connected by a (closed) 1-cell e in X , then e is their hull and $\overline{A B}$ is the homotopy class of e viewed as a path from A to B . Any $\alpha \in \pi_1(X, A_0)$ may be represented by a loop in the 1-skeleton $X^{(1)}$ of X traversing consecutively 0-cells $A_0, A_1, \dots, A_m = A_0$. Then $\alpha = \overline{A_0 A_1 \dots A_m}$, and we set

$$\psi(\alpha) = x_{A_0,A_1} x_{A_1,A_2} \cdots x_{A_{m-1},A_m} \in \Pi.$$

The right-hand side does not depend on the choice of the loop in $X^{(1)}$ representing α . Any two such loops may be related by the transformations inserting loops of type $\overline{A B A}$ where $A, B \in Y$ and loops of type $\overline{B_1 \dots B_m}$ where $B_1, B_2, \dots, B_m = B_1$ are consecutive 0-cells lying on the boundary of a 2-cell of X . The invariance of $\psi(\alpha)$ under these transformations follows from the equalities $x_{A,B} x_{B,A} = 1$ and

$$\prod_{i=0}^{m-1} x_{B_i, B_{i+1}} = 1$$

where we use that B_1, \dots, B_m lie in the same closed cell of X . It is clear that

$$\phi\psi(\alpha) = \phi\left(\prod_{i=0}^{m-1} x_{A_i, A_{i+1}}\right) = \prod_{i=0}^{m-1} \overline{A_0 A_i A_{i+1} A_0} = \overline{A_0 A_1 \dots A_m} = \alpha.$$

It is easy to deduce from the definitions that $\psi\phi = \text{id}$. Thus, ϕ and ψ are mutually inverse isomorphisms. \square

Since a straight CW space X is path-connected, the group $\pi_1(X, A)$ does not depend on the choice of $A \in X^{(0)}$ at least up to isomorphism. The isomorphisms in question can be chosen in a canonical way as follows. Given $A, B \in X^{(0)}$, we let $i_{A,B} : \pi_1(X, A) \rightarrow \pi_1(X, B)$ be the conjugation by $\overline{A B}$:

$$i_{A,B}(\alpha) = \overline{B A} \alpha \overline{A B} \in \pi_1(X, B) \quad \text{for } \alpha \in \pi_1(X, A).$$

Clearly, $i_{A,A} = \text{id}$ and $i_{B,A} = i_{A,B}^{-1}$ for any $A, B \in X^{(0)}$. For any $A, B, C \in X^{(0)}$, the automorphism $i_{C,A} i_{B,C} i_{A,B}$ of $\pi_1(X, A)$ is the conjugation by $\overline{A B C A}$.

7.2. A presentation of the dimer group. Consider a graph Γ and the dimer complex $X_{\mathcal{D}}$ where $\mathcal{D} = \mathcal{D}(\Gamma)$ is the set of perfect matchings in Γ .

Lemma 7.2. *All connected components of $X_{\mathcal{D}}$ are straight CW-spaces.*

Proof. Consider any $A, B \in \mathcal{D}$ lying in the same connected component of $X_{\mathcal{D}}$. By Lemma 6.4, the set $AB = (A \setminus B) \cup (B \setminus A)$ is finite. The proof of Lemma 6.4 shows that AB is a union of independent even cycles in Γ . Denote the set of these cycles by S . In the notation of Section 3.1, $[S] = AB$. As in the proof of Lemma 6.2, we observe that for all $T \subset S$, the set $[T]A$ is a perfect matching in Γ . Recall that the closed cells of $X_{\mathcal{D}}$ are embedded cubes. In particular, the based cube (A, S) determines a cube, Q , in $X_{\mathcal{D}}$ whose set of vertices contains both A and $B = [S]A$. We claim that Q is the hull of A, B . To prove this claim, it suffices to show that Q is a face of any cube Q' in $X_{\mathcal{D}}$ whose set of vertices contains A and B . Such a cube Q' can be represented by a based cube (A, S') such that $B = [T]A$ for some $T \subset S'$. We have $[T] = BA^{-1} = BA = AB$. Since a finite cyclic set of edges splits as a union of independent cycles in a unique way, $T = S$. So, Q is a face of Q' . \square

Lemma 7.2 allows us to apply the constructions of Section 7.1 to $X_{\mathcal{D}}$. This gives the isomorphisms between the dimer groups formulated at the end of Section 6.3.

We will denote the only edge of a perfect matching $A \in \mathcal{D}$ adjacent to a vertex v of Γ by A_v . A triple $A, B, C \in \mathcal{D}$ is said to be *flat* if for any vertex v of Γ , at least two of the edges A_v, B_v, C_v are equal.

Theorem 7.3. *The dimer group of any $A_0 \in \mathcal{D}$ is generated by the set $\{x_{A,B}\}_{A,B}$ where (A, B) runs over all ordered pairs of perfect matchings in Γ congruent to A_0 . The defining relations: $x_{A_0,A} = 1$ for each $A \in \mathcal{D}$ congruent to A_0 and $x_{A,C} = x_{A,B}x_{B,C}$ for each flat triple A, B, C of perfect matchings congruent to A_0 .*

Proof. The claim follows from Lemmas 6.4, 7.1, and 7.2. We need only to show that three perfect matchings $A, B, C \in \mathcal{D}$ are vertices of a cube in $X_{\mathcal{D}}$ if and only if the triple A, B, C is flat. Suppose that the triple A, B, C is flat. Then every vertex of Γ is incident to a unique edge belonging to at least two of the sets A, B, C . Such edges form a perfect matching, K , of Γ . For any vertex v of Γ , either $A_v = B_v = C_v = K_v$ or $A_v = B_v = K_v \neq C_v$ up to a permutation of A, B, C . In the first case, the sets AK, BK, CK contain no edges incident to v . In the second case, AK, BK contain no edges incident to v while CK contains two such edges C_v and K_v . This shows that AK, BK, CK are three pairwise independent cyclic sets of edges. They split (uniquely) as unions of independent cycles. All these cycles are even: their intersections with K are their halves. Denote the resulting sets of even cycles by X, Y, Z , respectively. Thus, $AK = [X], BK = [Y], CK = [Z]$. Equivalently, $A = [X]K$, $B = [Y]K$, and $C = [Z]K$. Then the cube in \mathcal{D} determined by the based cube $(K, X \cup Y \cup Z)$ contains $\{A, B, C\}$.

Conversely, suppose that $A, B, C \in \mathcal{D}$ are vertices of a cube in \mathcal{D} . It is easy to see that there is $K \in \mathcal{D}$ and a set of independent even cycles S with a partition $S = X \amalg Y \amalg Z$ such that $A = [X]K$, $B = [Y]K$, and $C = [Z]K$. If a vertex v of Γ is not a vertex of the cycles in S , then $A_v = B_v = C_v = K_v$. Pick any vertex $v \in \partial s$ with $s \in S$. If $s \in X$, then s is not a vertex of the cycles in Y and Z and $B_v = C_v = K_v$. The cases $s \in Y$ and $s \in Z$ are similar. In all cases, at least two of the edges A_v, B_v, C_v are equal. Therefore, the triple A, B, C is flat. \square

Corollary 7.4. *If Γ is a finite graph, then the dimer group of any $A_0 \in \mathcal{D} = \mathcal{D}(\Gamma)$ is presented by the generators $\{x_{A,B}\}_{A,B \in \mathcal{D}}$ subject to the relations $x_{A_0,A} = 1$ for each $A \in \mathcal{D}$ and $x_{A,C} = x_{A,B} x_{B,C}$ for each flat triple $A, B, C \in \mathcal{D}$.*

7.3. The fundamental groupoid. Consider a straight CW-space X and the fundamental groupoid $\pi_1(X, Y)$ where $Y = X^{(0)}$ is the 0-skeleton of X . This groupoid is a category whose objects are points of Y and whose morphisms are homotopy classes of paths in X with endpoints in Y . To make composition compatible with multiplication of paths, we write composition of morphisms in $\pi_1(X, Y)$ in the order opposite to the usual one. The arguments in the proof of Lemma 7.1 show that the groupoid $\pi_1(X, Y)$ can be presented by generators $\{x_{A,B} : A \rightarrow B\}_{A,B \in Y}$ subject to the relations $x_{A,C} = x_{A,B} x_{B,C}$ for any $A, B, C \in Y$ contained in a cell of X . The image of $x_{A,B}$ in $\pi_1(X, Y)$ is the morphism $\overline{AB} : A \rightarrow B$. As a consequence, in the setting of Section 7.2 the groupoid $\pi_1(X_{\mathcal{D}}, \mathcal{D})$ is presented by the generators $\{x_{A,B} : A \rightarrow B\}_{A,B}$ where (A, B) runs over all ordered pairs of congruent perfect matchings in Γ and the relations $x_{A,C} = x_{A,B} x_{B,C}$ for every flat triple A, B, C of congruent perfect matchings in Γ .

8. MATCHING GROUPS AND THEIR HOMOMORPHISMS

We define matching groups and study natural homomorphisms between them.

8.1. The matching groups. Recall that for a set A of edges of a graph Γ , we denote by ∂A the set of all vertices of Γ adjacent to at least one edge in A . Denote by Γ_A the subgraph of Γ whose vertices are the points of ∂A and whose edges are the edges of Γ with both endpoints in ∂A . In particular, all edges of Γ belonging to A are also edges of Γ_A . This gives a set of edges of Γ_A denoted A^p .

A matching A in Γ is a non-empty set of edges of Γ such that different edges in A have no common vertices. If A is a matching in Γ , then A^p is a perfect matching in Γ_A . The *matching group* of Γ at A is defined to be the dimer group of Γ_A at A^p :

$$\pi_A = \pi_A(\Gamma) = \pi_1(X_{\mathcal{D}}(\Gamma_A), A^p).$$

Computing this group from the 2-skeleton of $X_{\mathcal{D}}(\Gamma_A)$ in the standard way, we immediately see that this definition of π_A is equivalent to the one in the introduction. If A is a perfect matching in Γ , then $\Gamma_A = \Gamma$ and $A^p = A$ so that the matching group and the dimer group of A coincide.

The congruence of perfect matchings defined in Section 6.3 extends to arbitrary matchings in Γ as follows. Two matchings A, B are *congruent* if the set $AB = (A \setminus B) \cup (B \setminus A)$ is finite and $\partial A = \partial B$. These conditions imply that $\Gamma_A = \Gamma_B$ and that the perfect matchings A^p, B^p lie in the same connected component of the dimer complex $X_{\mathcal{D}}(\Gamma_A) = X_{\mathcal{D}}(\Gamma_B)$. Since this component is straight, Section 7.1 yields an isomorphism $i_{A,B} : \pi_A \rightarrow \pi_B$. We have $i_{A,A} = \text{id}$ and $i_{B,A} = i_{A,B}^{-1}$. If A, B, C are congruent matchings in Γ , then the automorphism $i_{C,A} i_{B,C} i_{A,B}$ of π_A is the conjugation by $\overline{ABCA} \in \pi_A$.

A matching A in Γ is *finite* if the graph Γ_A is finite (this condition is stronger than requiring A to be a finite set of edges). The matching group of a finite matching shares all group properties listed in Section 2.1 and Corollary 5.4, cf. Section 6.4.

8.2. Canonical homomorphisms. Let A, A' be matchings in a graph Γ such that $A' \subset A$. We construct a canonical homomorphism $j_{A',A} : \pi_{A'} \rightarrow \pi_A$ so that $j_{A,A} = \text{id}$ and $j_{A',A} j_{A'',A'} = j_{A'',A}$ for any sets $A'' \subset A' \subset A$.

Replacing, if necessary, Γ by Γ_A we can assume that A is a perfect matching. Let: E be the set of edges of Γ ; $G = 2^E$ be the power group of E ; \mathcal{D} be the set of perfect matchings in Γ . Let E' , G' , and \mathcal{D}' be similar objects associated with the subgraph $\Gamma' = \Gamma_{A'}$ of Γ . Clearly, $E' \subset E$, $G' \subset G$, every even cycle in Γ' is also an even cycle in Γ , and every cubic set of even cycles in Γ' is also a cubic set of even cycles in Γ . We define a map $j : X_{\mathcal{D}'} \rightarrow X_{\mathcal{D}}$ as follows. Set

$$C = A \setminus A' \subset E.$$

Any point $b \in X_{\mathcal{D}'}$ is represented by a triple $(B \in \mathcal{D}'$, a cubic set of glides $S \subset G'$, $x \in I^S$) such that $[T]B \in \mathcal{D}'$ for all $T \subset S$. Since $B \in \mathcal{D}'$, we have $\partial B = \partial A'$, $B \cap C = \emptyset$, and $B \cup C \in \mathcal{D}$. Let $j(b)$ be the point of X_G represented by the triple $(B \cup C, S, x)$. It is easy to check that $j(b)$ does not depend on the choice of (B, S, x) . Moreover, $j(b) \in X_{\mathcal{D}} \subset X_G$. Indeed, for $T \subset S$, we have $[T]B \in \mathcal{D}'$ and therefore $[T](B \cup C) = [T]B \cup C \in \mathcal{D}$. The resulting map $j : X_{\mathcal{D}'} \rightarrow X_{\mathcal{D}}$ is continuous and $j(A') = A$. The induced map in π_1 is the homomorphism $j_{A',A} : \pi_{A'} \rightarrow \pi_A$.

The homomorphisms $\{j_{A',A}\}$ are compatible with the homomorphisms $\{i_{A,B}\}$ of Section 8.1 as follows: if $A' \subset A$ and $B' \subset B$ are four matchings in Γ such that A is congruent to B and A' is congruent to B' , then the following diagram commutes:

$$\begin{array}{ccc} \pi_{A'} & \xrightarrow{i_{A',B'}} & \pi_{B'} \\ j_{A',A} \downarrow & & \downarrow j_{B',B} \\ \pi_A & \xrightarrow{i_{A,B}} & \pi_B \end{array}$$

If the graph Γ is finite, then the homomorphism $j_{A',A}$ is injective for any matchings $A' \subset A$ in Γ . This follows from Lemma 2.1 and the fact that $j : X' \rightarrow X$ is a local isometry; we leave the details to the reader.

9. MATCHINGS VS. BRAIDS

The braid groups of graphs share many properties of the matching groups, see [Ab], [CW]. We construct natural homomorphisms from the matching groups to the braid groups.

9.1. The braid groups. For a topological space P and an integer $n \geq 1$, the *ordered n -configuration space* $\tilde{C}_n = \tilde{C}_n(P) \subset P^n$ consists of all n -tuples of pairwise distinct points of P . The symmetric group S_n acts on \tilde{C}_n by permutations of the tuples, and the quotient $C_n = C_n(P) = \tilde{C}_n/S_n$ is the *unordered n -configuration space* of P . For $c \in C_n$, the fundamental group $\pi_1(C_n, c)$ is denoted $B_n(P, c)$ and called the *n -braid group of P at c* . The covering $\tilde{C}_n \rightarrow C_n$ determines (up to conjugation) a *permutation homomorphism* $\sigma_n : B_n(P, c) \rightarrow S_n$. Given a subspace Q of P , the obvious inclusion $C_n(Q) \subset C_n(P)$ induces the *inclusion homomorphism* $B_n(Q, c) \rightarrow B_n(P, c)$ for all $n \geq 1$ and $c \in C_n(Q)$.

The connection between configuration spaces and matchings stems from the following observation. Given a finite matching A in a graph Γ , one can consider the set \hat{A} consisting of the mid-points of the edges in A . This set is a point in the unordered N -configuration space $C_N(\Gamma)$ where $N = \text{card}(A) \geq 1$. To define \hat{A} , we tacitly assume that all edges of Γ are parametrized by $I = [0, 1]$. This allows us to consider the mid-points of the edges and, more generally, the convex combinations of points of the edges. The homomorphisms from the matching groups to the

braid groups defined below do not depend on the choice of parametrizations up to composition with transfer isomorphisms of braid groups along paths in $C_N(\Gamma)$.

9.2. V-orientations. We introduce a notion of a v-orientation of a graph Γ needed in our constructions. Given an even cycle s in Γ , the set of its vertices ∂s has a unique partition into two subsets called the *v-halves* such that the vertices of any edge in s belong to different v-halves. A *v-orientation* of Γ is a choice of a distinguished v-half in each even cycle in Γ .

9.3. The map Θ . Let Γ be a v-oriented finite graph admitting a perfect matching. Then Γ has $2N$ vertices for some $N \geq 1$ and all perfect matchings in Γ consist of N edges. The formula $A \mapsto \widehat{A}$ defines a map from the set \mathcal{D} of perfect matchings in Γ to $C_N(\Gamma)$. We extend this map to a continuous map Θ from the dimer space $X_{\mathcal{D}} = X_{\mathcal{D}}(\Gamma)$ to $C_N(\Gamma)$. The extension is based on the following idea: when a perfect matching A is glided along an even cycle s , the points of \widehat{A} not lying on the circle \underline{s} do not move while the points of $\widehat{A} \cap \underline{s}$ move to the points of $\widehat{sA} \cap \underline{s}$ along \underline{s} across the distinguished vertices of s . Here is a precise definition. A point $a \in X_{\mathcal{D}}$ is represented by a triple $(A \in \mathcal{D}, \text{ a finite set } S \text{ of independent even cycles in } \Gamma, x \in I^S)$ such that $[T]A \in \mathcal{D}$ for all $T \subset S$. Replacing, if necessary, A with tA for some $t \in S$ we can assume that $x(S) \subset [0, 1/2]$. Each edge $e \in A$ not lying on $\cup_{s \in S} s$ contributes its mid-point to $\Theta(a)$. Each edge $e \in A$ lying on a cycle $s \in S$ has an endpoint, a_e , belonging to the distinguished v-half of s and an endpoint, b_e , belonging to the complementary v-half of s . The edge e then contributes to $\Theta(a)$ the point

$$(1/2 + x(s))a_e + (1/2 - x(s))b_e \in e.$$

The coefficients are chosen so that for $x(s) = 0$ we get the mid-point of e , and for $x(s) = 1/2$ we get a_e . When e runs over A , we obtain an N -point set $\Theta(a) \subset \Gamma$. This gives a continuous map $\Theta : X_{\mathcal{D}} \rightarrow C_N(\Gamma)$ extending the map $\mathcal{D} \rightarrow C_N(\Gamma), A \mapsto \widehat{A}$.

9.4. The homomorphism θ_A . Let A be a finite matching in a v-oriented graph Γ (finite or infinite). The subgraph Γ_A of Γ defined in Section 8.1 is finite and has a perfect matching A^p . The v-orientation in Γ restricts to a v-orientation in Γ_A . Now, Section 9.3 yields a continuous map $\Theta : X_{\mathcal{D}}(\Gamma_A) \rightarrow C_N(\Gamma_A)$ where $N = \text{card}(A) \geq 1$. Let $\theta_A : \pi_A \rightarrow B_N(\Gamma, \widehat{A})$ be the composition

$$\pi_A = \pi_A(\Gamma) = \pi_1(X_{\mathcal{D}}(\Gamma_A), A^p) \xrightarrow{\Theta_*} \pi_1(C_N(\Gamma_A), \widehat{A^p}) = B_N(\Gamma_A, \widehat{A^p}) \rightarrow B_N(\Gamma, \widehat{A})$$

where the right arrow is the inclusion homomorphism. The homomorphism θ_A is not necessarily injective and may be trivial, see Example 9.6 below. The composition of θ_A with the permutation homomorphism $\sigma_N : B_N(\Gamma, \widehat{A}) \rightarrow S_N$ can be computed as follows. Mark the edges in A by the numbers $1, 2, \dots, N$. Any $\alpha \in \pi_A$ is represented by a sequence of consecutive glidings of A along certain even cycles s_1, \dots, s_n . Recursively in $i = 1, \dots, n$, we accompany the i -th gliding with the transformation of the marked matching which keeps the marked edges not belonging to s_i and replaces each marked edge in s_i with the adjacent edge in s_i sharing the vertex in the distinguished v-half of s_i and having the same mark. After the n -th gliding, we obtain the matching A with a new marking. The resulting permutation of the set $\{1, \dots, N\}$ is equal to $\sigma_N \theta_A(\alpha)$. Generally speaking, $\sigma_N \theta_A$ depends on the v-orientation of Γ , see Example 9.5 below. If A and B are congruent matchings

in Γ , then the canonical isomorphisms of matching groups and braid groups at A and B conjugate θ_A and θ_B .

Using θ_A one can construct braids from matchings in Γ : any tuple of matchings A_1, \dots, A_n congruent to A gives rise to the braid $\theta_A(\overline{AA_1 \cdots A_n A}) \in B_N(\Gamma, \hat{A})$.

9.5. Example. Consider the graph Γ in Figure 1 with vertices a, b, c, d, e, f . This graph has three cycles s_1, s_2, s_{12} formed, respectively, by the edges of the left square, the edges of the right square, all edges except the middle vertical edge. These cycles are even. We distinguish the v -halves, respectively, $\{a, e\}$, $\{c, e\}$, $\{b, d, f\}$. The vertical edges of Γ form a perfect matching A . The matching group $\pi_A = \pi_1(X_{\mathcal{D}}(\Gamma), A)$ is an infinite cyclic group with generator t represented by the sequence of slidings $A \mapsto s_1 A \mapsto s_{12} s_1 A \mapsto s_2 s_{12} s_1 A$. To compute $\sigma_3 \theta_A : \pi_A \rightarrow S_3$, we mark the edges in A with 1, 2, 3 from left to right. The transformations of A under the slidings are shown in Figure 1. Therefore $\sigma_3 \theta_A(t) = (231)$. The opposite choice of the distinguished v -half in s_2 gives $\sigma_3 \theta_A(t) = (213)$.

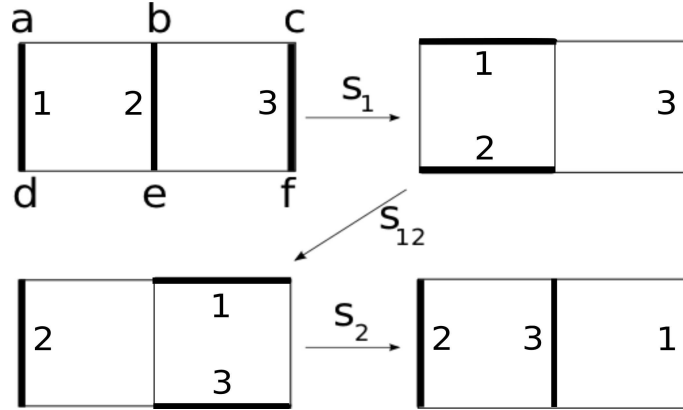


FIGURE 1. Transformations of a marked perfect matching

9.6. Example. Consider a finite bipartite graph Γ (the word “bipartite” means that the set of vertices is partitioned into two subsets V_0, V_1 such that every edge has one vertex in each). All cycles in Γ are even. We v -orient Γ by selecting in every cycle the v -half formed by the vertices in V_0 . For any matching A in Γ , we have $\Theta(X_{\mathcal{D}}(\Gamma_A)) \subset \prod_{v \in V_0 \cap \partial A} \Delta_v$ where Δ_v is the union of all half-edges of Γ_A adjacent to v . Since Δ_v is contractible, the map Θ is homotopic to a constant map. Hence $\theta_A = 1$. Other v -orientations in Γ may give non-trivial θ_A , cf. Example 9.5.

9.7. Generalization of θ_A . Let A be a finite matching in a v -oriented graph Γ and $N = \text{card}(A)$. The homomorphism $\theta_A : \pi_A \rightarrow B_N(\Gamma, \hat{A})$ can be included into a vast family of similar homomorphisms as follows. Let F be the set of edges of the (finite) graph Γ_A , i.e., the set of edges of Γ with both endpoints in ∂A . Pick a map $n : F \rightarrow \{0, 1, 2, \dots\}$ and set $|n| = \sum_{e \in F} n(e)$. Let Γ_A^n be the graph obtained from Γ_A by adding $2n(e)$ new vertices inside each edge e . The canonical homeomorphism $X_{\mathcal{D}}(\Gamma_A) \approx X_{\mathcal{D}}(\Gamma_A^n)$ (cf. Remark 6.6.1) induces an isomorphism $\pi_A(\Gamma) \simeq \pi_{A_n}(\Gamma_A^n)$ where A_n is the perfect matching in Γ_A^n induced by $A^p \in \mathcal{D}(\Gamma_A)$. The v -orientation of Γ induces a v -orientation of Γ_A^n : the distinguished v -half of a cycle in Γ_A^n is the

one including the distinguished v-half of the corresponding cycle in Γ . We define a homomorphism $\theta_A^n : \pi_A(\Gamma) \rightarrow B_{N+|n|}(\Gamma, \widehat{A}_n)$ as the composition

$$\pi_A(\Gamma) \simeq \pi_{A_n}(\Gamma_A^n) \xrightarrow{\theta_{A_n}} B_{N+|n|}(\Gamma_A^n, \widehat{A}_n) \rightarrow B_{N+|n|}(\Gamma, \widehat{A}_n)$$

where the right arrow is the inclusion homomorphism. For $n = 0$, we have $\theta_A^n = \theta_A$. In Example 9.6 the graph Γ^n is bipartite and so $\theta_A^n = 1$ for all n .

Composing θ_A^n with $\sigma_{N+|n|}$ we obtain a family of homomorphisms from the matching groups to the symmetric groups. Is it true that $\bigcap \bigcap_n \text{Ker}(\sigma_{N+|n|}\theta_A^n) = 1$ where the first intersection runs over all v-orientations of the graph?

10. THE SPACE OF DIMER LABELINGS

We interpret the dimer complex of a finite graph in terms of so-called dimer labelings. This determines a canonical embedding of the dimer complex into a cube. We first define a certain evaluation map in a more general setting.

10.1. The evaluation map. Assume that the power group $G = 2^E$ of a set E is equipped with a gliding system such that any two independent glides are disjoint as subsets of E . Consider the associated cubed complex X_G . Assigning to each set $A \subset E$ its characteristic function $\delta_A : E \rightarrow \{0, 1\} \subset I$ we obtain a map from the 0-skeleton $X_G^{(0)} = G$ of X_G to I^E . This map extends to a map $\omega : X_G \rightarrow I^E$ by

$$\omega(a) = \delta_A + (1 - 2\delta_A) \sum_{s \in S} x(s) \delta_s : E \rightarrow I$$

for any point $a \in X_G$ represented by a triple $(A \in G, \text{a cubic set of glides } S \subset G, \text{ a map } x : S \rightarrow I)$. By the assumption on the gliding system, different elements of S are disjoint as subsets of E . Therefore, for any $e \in E$,

$$(10.1.1) \quad \omega(a)(e) = \begin{cases} \delta_A(e) & \text{if } e \in E \setminus [S] = E \setminus \bigcup_{s \in S} s \\ x(s) & \text{if } e \in s \setminus A \text{ with } s \in S \\ 1 - x(s) & \text{if } e \in s \cap A \text{ with } s \in S. \end{cases}$$

This formula implies that the map $\omega : X_G \rightarrow I^E$ is well-defined, continuous, and its restriction to any cube in X_G is injective. We call ω the *evaluation map*.

Lemma 10.1. *Suppose that the gliding system in G is such that for any different cubic sets of glides $S_1, S_2 \subset G$, we have $[S_1] \neq [S_2]$. Suppose that a set $\mathcal{D} \subset G$ satisfies the following condition: any glide $s \subset E$ has a partition into two non-empty subsets such that if $A \in \mathcal{D}$ and $sA \in \mathcal{D}$, then $s \cap A$ is one of these subsets. Then the restriction of $\omega : X_G \rightarrow I^E$ to $X_{\mathcal{D}} \subset X_G$ is injective.*

Proof. Assume that $\omega(a_1) = \omega(a_2)$ for some $a_1, a_2 \in X_{\mathcal{D}}$. We show that $a_1 = a_2$. Let us represent each a_i by a triple $(A_i \in \mathcal{D}, S_i, x_i : S_i \rightarrow I)$ as above. Passing, if necessary, to a face of the cube, we can assume that $0 < x_i(s) < 1$ for all $s \in S_i$. Set $f_i = \omega(a_i) : E \rightarrow I$. It follows from (10.1.1) that $f_i^{-1}((0, 1)) = \bigcup_{s \in S_i} s = [S_i]$. The equality $f_1 = f_2$ implies that $[S_1] = [S_2]$. By the assumptions of the lemma, $S_1 = S_2$. Set $S = S_1 = S_2$. We prove below that for any $s \in S$, either

- (*) $s \cap A_1 = s \cap A_2$ and $x_1(s) = x_2(s)$ or
- (**) $s \cap A_1 = s \setminus A_2$ and $x_1(s) + x_2(s) = 1$.

For an s of the second type, replace (A_2, S, x_2) with $(A'_2 = sA_2, S, x'_2)$ where $x'_2 : S \rightarrow I$ carries s to $1 - x(s)$ and is equal to x_2 on $S \setminus \{s\}$. The triple (A'_2, S, x'_2) represents the same point $a_2 \in X_{\mathcal{D}}$ and satisfies $s \cap A_1 = s \cap A'_2$ and $x_1(s) = x'_2(s)$.

Since different $s \in S$ are disjoint as subsets of E , such replacements along different s of the second type do not interfere with each other and commute. Proceeding in this way, we obtain a new triple (A_2, S, x_2) representing a_2 and satisfying $(*)$ for all $s \in S$. Then $x_1 = x_2 : S \rightarrow I$. Also,

$$A_1 \cup \bigcup_{s \in S} s = f_1^{-1}((0, 1]) = f_2^{-1}((0, 1]) = A_2 \cup \bigcup_{s \in S} s.$$

Since different $s \in S$ are disjoint as subsets of E and satisfy $s \cap A_1 = s \cap A_2$, we deduce that $A_1 = A_2$. Thus, $a_1 = a_2$.

It remains to prove $(*)$ or $(**)$ for each $s \in S$. Set $f = f_1 = f_2 : E \rightarrow I$. By (10.1.1), the map f takes at most two values on $s \subset E$. Suppose first that $f|_s$ takes two distinct values (whose sum is equal to 1). By (10.1.1), f is constant on both $s \cap A_i$ and $s \setminus A_i$ for $i = 1, 2$. If $f(s \cap A_1) = f(s \cap A_2)$, then we have $(*)$. If $f(s \cap A_1) = f(s \setminus A_2)$, then we have $(**)$. Suppose that f takes only one value on s . The assumption on the set \mathcal{D} implies that $s \cap A_1 \neq \emptyset$ and $s \setminus A_1 \neq \emptyset$. Since $f = \omega(a_1)$ takes only one value on s , formula (10.1.1) implies that this value is $1/2$. Then $x_1(s) = x_2(s) = 1/2$. The assumption on \mathcal{D} implies that either $s \cap A_1 = s \cap A_2$ or $s \cap A_1 = s \setminus A_2$. This gives, respectively, $(*)$ or $(**)$. \square

10.2. Dimer labelings. Consider a finite graph Γ with the set of edges E . A *dimer labeling* of Γ is a labeling of the edges of Γ by non-negative real numbers such that for every vertex of Γ , only one or two of the labels of the adjacent edges are non-zero and their sum is equal to 1. The set $L = L(\Gamma)$ of dimer labelings of Γ is a closed subset of the cube I^E and is endowed with the induced topology. The characteristic function of a perfect matching in Γ is a dimer labeling. In this way, we identify the set of perfect matchings $\mathcal{D} = \mathcal{D}(\Gamma)$ of Γ with a subset of L .

Theorem 10.2. (i) *The set $\mathcal{D} \subset L$ lies in a path-connected component, L_0 , of L .*
(ii) *Let $\omega : X_G \rightarrow I^E$ be the evaluation map from Section 10.1 where $G = 2^E$ carries the even-cycle gliding system. The restriction of ω to the dimer complex $X_{\mathcal{D}} \subset X_G$ is a homeomorphism of $X_{\mathcal{D}}$ onto L_0 .*
(iii) *All other components of L are homeomorphic to the dimer complexes of certain subgraphs of Γ .*

Proof. It follows from the definitions that $\omega(X_{\mathcal{D}}) \subset L$ and that the restriction of ω to $\mathcal{D} \subset X_{\mathcal{D}}$ is the identity map $\text{id} : \mathcal{D} \rightarrow \mathcal{D}$. We apply Lemma 10.1 to $\mathcal{D} \subset G$. The condition on the gliding system holds because a subset of E cannot split as a union of independent cycles in two different ways. The partition of even cycles into halves satisfies the condition on \mathcal{D} . By Lemma 10.1, the restriction of ω to $X_{\mathcal{D}}$ is injective. Since $X_{\mathcal{D}}$ is path connected, $\omega(X_{\mathcal{D}})$ is contained in a path connected component, L_0 , of $L(\Gamma)$. By the above, $\mathcal{D} \subset \omega(X_{\mathcal{D}}) \subset L_0$. This proves (i).

We claim that $\omega(X_{\mathcal{D}}) = L_0$. Indeed, pick any dimer labeling $\ell : E \rightarrow I$ in L_0 and prove that $\ell \in \omega(X_{\mathcal{D}})$. It is clear that the set $\ell^{-1}((0, 1)) \subset E$ is cyclic. It splits uniquely as a disjoint union of $n \geq 0$ independent cycles s_1, \dots, s_n . If s_i is odd for some i , then $\ell|_{s_i} = 1/2$: otherwise, s_i could be partitioned into edges with $\ell < 1/2$ and edges with $\ell > 1/2$ and would be even. Then, any deformation of ℓ in $L(\Gamma)$ must preserve $\ell(s_i)$. This contradicts the assumption that ℓ can be connected by a path in L to an element of \mathcal{D} . Therefore the cycles s_1, \dots, s_n are even. For each $i = 1, \dots, n$, pick a half, s'_i , of s_i . The definition of a dimer labeling implies that ℓ takes the same value, $x_i \in (0, 1)$, on all edges belonging to s'_i and the value $1 - x_i$ on all edges in $s_i \setminus s'_i$. Clearly, the edges belonging to the set $A = \ell^{-1}(\{1\}) \cup \bigcup_{i=1}^n s'_i \subset E$

have no common vertices. Since each vertex of Γ is incident to an edge with positive label, it is incident to an edge belonging to A . Therefore $A \in \mathcal{D}$. Since s_1, \dots, s_n are independent even cycles and $A \cap s_i = s'_i$ for each i , all vertices of the based cube $(A, S = \{s_1, \dots, s_n\})$ belong to \mathcal{D} . The triple $(A, S, x : S \rightarrow I)$, where $x(s_i) = 1 - x_i$ for all i , represents a point $a \in X_{\mathcal{D}}$ such that $\omega(a) = \ell$. So, $\ell \in \omega(X_{\mathcal{D}})$.

We conclude that the map $\omega|_{X_{\mathcal{D}}} : X_{\mathcal{D}} \rightarrow L_0$ is a continuous bijection. Since $X_{\mathcal{D}}$ is compact and L_0 is Hausdorff, this map is a homeomorphism. This proves (ii).

Arbitrary components of the space $L(\Gamma)$ can be described as follows. Consider a set C of independent odd cycles in Γ . Deleting from Γ the vertices of these cycles and all the edges incident to these vertices, we obtain a subgraph Γ^C of Γ . Each dimer labeling of Γ^C extends to a dimer labeling of Γ assigning $1/2$ to the edges belonging to the cycles in C and 0 to all other edges of Γ not lying in Γ^C . This defines an embedding $i_C : L(\Gamma^C) \hookrightarrow L(\Gamma)$. The arguments above show that the image of i_C is a union of connected components of $L(\Gamma)$. In particular, $i_C(L_0(\Gamma^C))$ is a component of $L(\Gamma)$. Moreover, every component of $L(\Gamma)$ is realized as $i_C(L_0(\Gamma^C))$ for a unique C . (In particular, $L_0(\Gamma)$ corresponds to $C = \emptyset$.) It remains to note that $L_0(\Gamma^C)$ is homeomorphic to the dimer complex of Γ^C . \square

Corollary 10.3. *All connected components of $L(\Gamma)$ are aspherical and are homeomorphic to non-positively curved cubed complexes.*

11. EXTENSION TO HYPERGRAPHS

We extend the definition of dimer groups and matching groups to hypergraphs.

11.1. Hypergraphs. By a *hypergraph* we mean a triple $\Gamma = (E, V, \partial)$ consisting of two sets E, V and a map $\partial : E \rightarrow 2^V$ such that $\bigcup_{e \in E} \partial e = V$ and $\partial e \neq \emptyset$ for all $e \in E$. The elements of E are *edges* of Γ , the elements of V are *vertices* of Γ , and ∂ is the *boundary map*. For $e \in E$, the elements of $\partial e \subset V$ are the *vertices of e* .

We briefly discuss a few examples. A graph gives rise to a hypergraph in the obvious way. Every matrix M over an abelian group yields a hypergraph whose edges are non-zero rows of M , whose vertices are columns of M , and whose boundary map carries a row to the set of columns containing non-zero entries of this row. A CW-complex gives rise to a sequence of hypergraphs associated as above with the matrices of the boundary homomorphisms in the cellular chain complex. Every hypergraph (E, V, ∂) determines the *dual hypergraph* (V, E, ∂^*) where $\partial^*(v) = \{e \in E \mid v \in \partial(e)\}$ for all $v \in V$.

11.2. The gliding system. Given a hypergraph $\Gamma = (E, V, \partial)$, we call two sets $s, t \subset E$ *independent* if $\partial s \cap \partial t = \emptyset$. Of course, independent sets are disjoint.

A *cyclic set of edges* in Γ is a finite set $s \subset E$ such that for every $v \in V$, the set $\{e \in s \mid v \in \partial e\}$ has two elements or is empty. A cyclic set of edges is a *cycle* if it does not contain smaller non-empty cyclic sets of edges.

Lemma 11.1. *If $s \subset E$ is a cyclic set of edges, then the cycles contained in s are pairwise independent and s is their disjoint union.*

Proof. Define a relation \sim on s by $e \sim f$ if $e, f \in s$ satisfy $\partial e \cap \partial f \neq \emptyset$. This relation is reflexive and symmetric but possibly non-transitive. It generates an equivalence relation on s ; the corresponding equivalence classes are the cycles contained in s . This implies the lemma. \square

A cycle $s \subset E$ is *even* if s has a partition into two subsets called the *halves* such that the edges belonging to the same half have no common vertices. It is easy to see that if such a partition $s = s' \cup s''$ exists, then it is unique and $\cup_{e \in s'} \partial e = \cup_{e \in s''} \partial e$.

Even cycles as glides with the independence relation above form a regular gliding system in the power group $G = 2^E$. By Corollary 4.5, the associated cubed complex $X_G = X_G(\Gamma)$ (with 0-skeleton G) is nonpositively curved. As in Section 6.2, a choice of a distinguished element in each even cycle determines an orientation of G and a typing homomorphism $\mu_A : \pi_1(X_G, A) \rightarrow \mathcal{A}(G)$ for $A \in G$. If E is finite, then the right-angled Artin group $\mathcal{A}(G)$ is finitely generated and μ_A is an injection.

11.3. Matchings. A *matching* in a hypergraph $\Gamma = (E, V, \partial)$ is a set $A \subset E$ such that $\partial e \cap \partial f = \emptyset$ for any distinct $e, f \in A$. A matching A is *perfect* if $\cup_{e \in A} \partial e = V$.

Let $\mathcal{D} = \mathcal{D}(\Gamma)$ be the set of all perfect matchings of Γ . The same arguments as in the proof of Lemma 6.2 show that $\mathcal{D} \subset G = 2^E$ satisfies the square condition and the cube condition. The associated cubed complex $X_{\mathcal{D}} \subset X_G$ is the *dimer complex* of Γ . Both X_G and $X_{\mathcal{D}}$ are nonpositively curved. For $A \in \mathcal{D}$, the group $\pi_1(X_{\mathcal{D}}, A)$ is the *dimer group* of Γ at A . If the sets E and V are finite, then $X_{\mathcal{D}}$ is connected; its fundamental group is the *dimer group* of Γ .

Given a matching $A \subset E$, we define a hypergraph $\Gamma_A = (E_A, V_A, \partial_A)$ where $V_A = \cup_{e \in A} \partial e$, $E_A = \{e \in E \mid \partial e \subset V_A\}$, and ∂_A is the restriction of ∂ to E_A . Clearly, $A \subset E_A$ is a perfect matching in Γ_A . The group $\pi_A = \pi_1(X_{\mathcal{D}}(\Gamma_A), A)$ is the *matching group* of Γ at A .

With these definitions, all the results above concerning the dimer groups and the matching groups in graphs extend to hypergraphs with appropriate changes. We leave the details to the reader.

APPENDIX A. TYPING HOMOMORPHISMS RE-EXAMINED

We state several properties of the typing homomorphisms defined in Section 5.2. Let: Γ be a graph with the set of edges E ; $G = 2^E$ be the power group of G ; and $\mathcal{D} \subset G$ be the set of perfect matchings of Γ . By Section 6.2, a choice of an element $e_s \in s$ in each even cycle s in Γ determines an orientation on G . For $A \in G$, consider the corresponding typing homomorphism $\mu_A : \pi_1(X_G, A) \rightarrow \mathcal{A}$ where $\mathcal{A} = \mathcal{A}(G)$ is the right-angled Artin group associated with the even-cycle gliding system in G . Composing μ_A with the inclusion homomorphism $\pi_A = \pi_1(X_{\mathcal{D}}, A) \rightarrow \pi_1(X_G, A)$ we obtain a homomorphism $\pi_A \rightarrow \mathcal{A}$ also denoted μ_A . The same homomorphism is obtained by restricting the orientation on G to \mathcal{D} and taking the associated typing homomorphism. If E is finite, then $\mu_A : \pi_A \rightarrow \mathcal{A}$ is an injection.

If $e_s^1, e_s^2 \in s$ belong to the same half of s for each even cycle s in Γ , then the families $\{e_s^1 \in s\}_s$ and $\{e_s^2 \in s\}_s$ determine the same orientation on G and on \mathcal{D} . Thus, to specify an orientation on \mathcal{D} and the typing homomorphism $\mu_A : \pi_A \rightarrow \mathcal{A}$ it suffices to specify a half in each s . When the distinguished half of an even cycle s_0 is replaced with the complementary half, the homomorphism μ_A is replaced by its composition with the automorphism of \mathcal{A} inverting the generator $g_{s_0} \in \mathcal{A}$ and fixing the generators $g_s \in \mathcal{A}$ for $s \neq s_0$.

Let \mathcal{B} be the right-angled Artin group with generators $\{h_e\}_{e \in E}$ and relations $h_e h_f = h_f h_e$ for all edges $e, f \in E$ having no common vertices. Choose in each even cycle s in Γ a half $s' \subset s$ and consider the associated typing homomorphism $\mu_A : \pi_A \rightarrow \mathcal{A}$ with $A \in \mathcal{D}$. The formula $u(g_s) = \prod_{e \in s \setminus s'} h_e^{-1} \prod_{e \in s'} h_e$ defines a homomorphism $u : \mathcal{A} \rightarrow \mathcal{B}$. We claim that $u\mu_A = 1$. To see this, use (5.2.1) to

compute μ_A on a path α in $X_{\mathcal{D}}$ formed by $n \geq 0$ consecutive 1-cells e_1, \dots, e_n . If $A_k \in \mathcal{D}$ is the terminal endpoint of e_k and the initial endpoint of e_{k+1} , then

$$u(g_{|e_k|}^{\nu_k}) = \prod_{e \in A_{k-1}} h_e^{-1} \prod_{e \in A_k} h_e.$$

Multiplying over $k = 1, \dots, n$, we obtain $u\mu_A(\alpha) = \prod_{e \in A_0} h_e^{-1} \prod_{e \in A_n} h_e$. For $A_0 = A_n = A$, this gives $u\mu_A(\alpha) = 1$.

The homomorphism $\mu_A : \pi_A \rightarrow \mathcal{A}$ induces a homomorphism in cohomology (with any coefficients) $\mu_A^* : H^*(\mathcal{A}) \rightarrow H^*(\pi_A)$. This may give non-trivial cohomology classes of π_A . The algebra $H^*(\mathcal{A})$ can be computed from the fact that the cells of the Salvetti complex appear in the form of tori and so, the boundary maps in the cellular chain complex are zero (see, for example, [Ch]). The equality $u\mu_A = 1$ above implies that μ_A^* annihilates $u^*(H^*(\mathcal{B})) \subset H^*(\mathcal{A})$.

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